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How Visualization Can Improve the Understanding of Mathematics

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Abstract

High school students have a difficult time taking the new End-Of-Course exams in math. Why are students having this difficulty? This could be because students lack the understanding of visualization in mathematics. Visualization can be helpful to certain students who think this way, while also confusing to those same students, and others as well. However, to what extent do high stakes tests, such as the End-Of-Course exams in Algebra I and II, incorporate mathematical visualization? How can visualization help improve the understanding of mathematics? This review will explain the advantages and disadvantages of incorporating visualization in these exams, how students and teachers can learn to understand visual techniques within math education, and how students approach mathematical situations mentally and physically

What is Visualization exactly? Especially in terms of visualization, we believe it only means in what we can see, when however, it could also be what we cannot see. What can very well be unseen is within data representations. Visualization can give a better understanding to mathematics and could increase the number of students within the field if incorporated more into a classroom setting. *The Role of Visual Representations in the Learning of Mathematics* by Arcavi states that “the visual display of information enables us to 'see' the story, to envision some cause-effect relationships, and possibly to remember it vividly” (Arcavi, 2003).

The definition of visual images is also self-explanatory. Norma Presmeg explained in *Visualisation in High School Mathematics* that “a visual image is a mental scheme depicting visual or spatial information” (Presmeg, Visualization in High School Mathematics, 1986); however the definition can be broad to include different kinds of imagery. The methods to solve mathematical situations are visual, which involves imagery with or without a graph and nonvisual which doesn’t require visualization at all.

There are other definitions in regards to visualization. Mathematical Visuality is the extent that students use visualization. There are visualizers and nonvisualizers, which are students that either prefer to use visual methods or not. These same definitions can also be used for teachers as well.

One difficulty student’s face from using visual methods is that it becomes time-consuming as well as the visualizers having trouble with terminology.

According to Bartlett, within Presmeg’s article of *Visualization in High School Mathematics*, there are limitations, some effectiveness and advantages of imagery. Any kind of vivid imagery is considered an advantage while concrete imagery is considered effective with the use of some nonvisual formulae. Dynamic imagery is effective even though it was not frequently used in studies. Imagery serving as an abstract function is also effective in two ways: concretising the referent and pattern imagery.

In regards to teaching methods in high school mathematics, it was found that teachers were classified in groups just like students but by how they taught either visually, some visual techniques, and non-visually. To summarize, teachers who had a nonvisual method taught a more lecturing style in their classrooms. As for the visual teachers and the “middle” group of teachers, they also taught a lecture style but they also applied scenarios from the real world and creativity. However, teachers regardless of their teaching methods need to be aware of the visual difficulties of generalization.

When it comes to mathematics education, it doesn’t take much to succeed in high school mathematics without learning visualization. Within *Visualization in High School Mathematics,* Krutetskii concluded that verbal-logical thinking relates to a student’s mathematical skills while visual-spatial thinking relates to the type of student. It is important for teachers to identify which students are visualizers in order to incorporate different teaching methods. Presmeg published another article, *Visual and Mathematical Giftedness,* that classified these students into two different groups according to their learning styles.

In the *Visual and Mathematical Giftedness* article, most high school students that are considered “stars” recommended by their teachers are classified as non-visualizers in mathematics.

Within the study, three categories were identified. The study came up with two definitions for visualizers and non-visualizers. Visualizers used both memorization of formulas as well as an imagination or visual imagery to go about a problem while a non-visualizer uses only the memorization of formulas and no visual imagery to solve a problem.

In a study scoring 277 students or "pupils" during a Mathematical Visuality score, 59 pupils out of 277 ranged from 20-33, where 0 is closer to non-visual and 36 is visual. Though the average is 18, majority of the students were closer to being classified as visualizers. It was from this conclusion at that all the "stars" were non-visualizers.

As to why this is, Krutetskii's work within the article, *Visual and Mathematical Giftedness*, answered the question. However Moses, another analyzer, summed up the work into these three categories:

"(1) subjects for whom the visual element is dominant: abstract problems are solved visually;

(2) subjects for whom the abstract element is dominant: geometry problems are solved logically, abstractly;

(3) subjects for whom the two elements are in equilibrium" (Presmeg, Visualization and Mathematical Giftedness, 1986)

The main point of Krutetskii's research is the level of math abilities which is determined by verbal-logical components and mathematical giftedness which is determined by visual-pictorial components. However, Krutetskii excluded the components of mathematical giftedness like swift mental processing, memory of formulas, symbols and numbers, computational abilities, ability for spatial concepts, and an ability to visualize abstract mathematical relationships and dependencies though they are considered very useful.

There are internal and external factors that influence the reason why the “stars” in high school are non-visualizers. One internal factor is the time factor. Many of these students have memorized formulas over an interval of time. Another internal factor is their particular way of thinking. When first taught, the students only had to visualize the problem or formula once then after a while, they “got used to it” (Presmeg, Visualization and Mathematical Giftedness, 1986).

However, vivid imagery has actually been quite effective for high school mathematics

“(1) vivid images of all types were found to have mnemonic advantages;

(2) an effective combination was found to be concrete imagery alternating with abstract nonvisual modes such as analysis, logic or a facile nonvisual use of formulae;

(3) although used only infrequently in this study, dynamic imagery was found to be potentially effective; and

(4) imagery which served an abstract function was found to be effective” (Presmeg, Visualization and Mathematical Giftedness, 1986).

However Krutetskii recognized, some of the limitations associated with the one- case concreteness of imagery can be overcome, so this imagery can be made to serve an abstract function in mathematics. In conclusion of imagery, it can be used in two ways: pattern imagery, where pure relationships are depicted and are more important than concrete detail, or by concretising the referent.

Internal factors are dependent of external factors and there are three external factors that can influence non-visualizers to excel in mathematics.

A stereotype that most students believe is that mathematics actually favors non-visualizers because in most cases being visual is not necessary. Many mathematicians either thought algebraically or geometrically. Another factor is the school curriculum where the time constraints in tests and exams do not favor the visual thinkers especially since most problems are time-consuming because you need to use visual methods to solve it. This can also be a result from the teaching methods, which is a third factor. Most teachers use visual examples but they give more emphasis on using non-visual methods, which can also derive from the use of textbooks. With the emphasis in this teaching method, many visual thinking students believed that in order to succeed in math, they would have to memorize formulas and many did not want to pursue a future in math. However, visualizers did better in classes where the teacher stressed the art of abstraction and generalization with the use of “pattern imagery and the use of curtailed methods” (Presmeg, Visualization and Mathematical Giftedness, 1986). Had teachers been aware of these difficulties and known that they could be resolved, maybe visual thinking students could be placed in the category of “stars”.

Kevin is a perfect example of being a visualizing “star”. However, though he is an active student when it comes to his thoughts about teaching math, his environment considers him to be a pupil “with problems”.

Using the scores of the WSAT “which measures the student’s spatial ability to rotate two-dimensions figures (Wheatley Spatial Ability Test -Wheatley, 1978)” (Cruz, Febles, & Diaz, 2000) a research the study was conducted on students who may or may not use the visualization process. By his high scores on the WSAT, Kevin was selected for the research study. He was a special pupil at the age of fourteen who scores a 90.5 out of a 100.

Based upon his pedagogic beliefs, Kevin believes that the basic classroom problems are routine and systematic, “taught and explained”, but he sees that not all problems can be solved by using only one routine method and anyone can solve a problem “according to his method”.

After making a model of a box, the study recognized that Kevin is a kinesthetic and dynamic imagery because of his doubt and constant rotation of the model and his mental image of it in his mind. When it came to Kevin solving three problems using the proposed equation system, he left the first part of the exam blank because, according to his teacher, hasn’t acquired the algebraic tools. However, when he solved the verbal question, Kevin used visual methods like concrete imagery and logical reasoning. In another problem, Kevin used algebra but kept mental notes of unknown variables. When it came to solving the ducks and rabbits problem, he used pure mental imagery without a single numeric strategy, but like most of the problems he had to solve, he used trial and error. In conclusion, his teacher failed him on the exam even though he answered every single one correctly. It is because the teacher was looking for algebraic strategies not visual methods.

In conclusion, Kevin didn’t pass mathematics nor pass the final exam with the addition to frequent absences. However, studying on Kevin has also recognized the importance of noticing visual methods on student’s work. The standard form of teaching is considered the norm and is what fails pupils because the visual method is neglected and teachers also tend to not consider the knowledge acquired outside of the school setting.

With these notions recognized, the US Curricular Standards encourages visualization as a fundamental part of understanding two and three-dimensional figures. The article points out the following problems that visualization can fix which is to stimulate visualization as a tool for teaching methods, also encourage other students to value their classmates gift of using visualization, and to develop visual education within the classroom. However, that can be a difficulty when it is always the students' decision on which method they would prefer to use to solve mathematical situations.

When it comes to a decision to solve a situation algebraically or visually, students prefer to use algebra. *On the Reluctance to Visualize in Mathematics* "…students seem to choose a symbolic framework to process mathematical information rather than a visual one" (Eisenberg and Dreyfus [7]). This problem probably arose from students' basic understanding of mechanics and the lack of connections between visual and analytic concepts.

It is in our mindset that we believe nonvisual techniques are the best way to solve a situation no matter if there is a visual underlining. Some mathematicians believe that higher mathematics requires more of a nonvisual approach.

Most students only use one way of representation, however using more than one representation and the transitions between them is the best way to answer mathematical situations. There are three main components to understand a function according to Schwarz within *On Understanding How Students Learn to Visualize Function Transformations* by Eisenberg and Dreyfus "The ability to pass from one representation in solving a problem, the flexibility to use the appropriate one, and also to see one representation when working in another" (Eisenberg & Dreyfus, On Understanding How Students Learn to Visualize Function Transformations). Also another contribution to understanding functions is the ability to visualize these transformations.

However visualization has its difficulties as Presmeg pointed out classifying students as visualizers and non-visualizers. Students who were classified as visualizers are unaware of transitions between images and that the school system only enforces non-visualization methods. As a result, advanced students tend to use only non-visualization methods even though the mathematical situation can be more complicated. This can be because that these students have not “constructed cognitive frameworks” (Eisenberg & Dreyfus, On Understanding How Students Learn to Visualize Function Transformations).

From what Eisenberg and Dreyfus learned, visualization should be an important part of learning, but because it isn’t a standard method there is little opportunity in teaching it. There are two difficulties that limit students understanding the visual approach. One is the function transformation itself and other is the instructional format. However, when it came to visual approach, some students were able to use transformations that were easier than others.

In conclusion to this study, it is difficult for students to understand the transition of function transformations. The complexity of a situation determines a student’s ability to use visualization and transform a function, however visualization, in this manner, can be taught.

Visualization, however, can be used as a symbolic context which is where visualization can be used in three different ways for mathematical students: as a support or proof, a way to resolve the solution, or as a way to see what the hidden answer is which most might probably miss. “Visualization may function as a tool to extricate oneself from situations in which one may be uncertain about how to proceed” (Arcavi, 2003).

It is in fact that visualization can encourage verbalization and also exclude it from any problem which is why it should be accepted into the learning practice with algebraic reasoning.

What can be seen by mathematical students can be missed by experts when using visual concepts. “Moreover, it also implies the competence to disentangle contexts in which similar objects can mean very different things, even to the same expert” (Arcavi, 2003).

However, visualization within mathematics education can provide some difficulties when trying to be accepted. Within education there is a “cultural” difficulty which means the values within mathematics have controversies on what is acceptable or not. Some students and teachers regard visualization as a “risky” procedure and can be rejected. Another difficulty is the relationship between visual and analytic representations. The back and forth motion between to the two in order to solve situations “can be a long-winded, context dependent, non- linear and even tortuous process for students (e.g. Schoenfeld, Smith and Arcavi, 1993)”(Arcavi, 2003). There is also a sociological problem which is the actual environment of the classrooms where there is much diversity and those of higher standing would view visualization taught in the classroom to be a ‘deficit’. However, with motivational visualization practices in the classroom, visualization can actually help students more and keep them interested in the field of math.

This practice of administering visualization more in math education could start in the early on stages of Pre-Calculus which consists of Algebra and Plane Trigonometry. If students cannot understand the visualization used in pre-calculus then there likeness for math diminishes, making it unlikely for students to continue taking higher math courses.

Though pre-calculus is taught at all school levels from high school to universities, those who successfully take it don’t continue with the next math course to take Calculus I. Pre-calculus helps students prepare and understand all the properties it takes to take Calculus I. From Arizona State University, “23% who passed pre-calculus at the university level completed Calculus I, but

5% of those students who passed Calculus I go on to take Calculus II" as stated in The Precalculus Concept Assessment: A tool For Assessing Studens' reasoning Abilities and Understandings (Carlson, Oehrtman, & Engelke). As a result of this study, this is a major reason why students choose to leave the STEM profession which reflects our own economy.

Though factors contributing to the decline of studying Calculus is not known or understood, there are multiple events that happen to students that make them decide to stray away from pursuing a career in the STEM field. From Seymour, "poor teaching by faculty, course pace, course load, inadequate academic advising, financial problems, and language barriers" (Carlson, Oehrtman, & Engelke) are part of the many events that can influence students.

Since there are no standards or a basic foundation for all instructors and developers to agree upon as to what has to be taught in pre-calculus, teachers use their own course curriculum and opinions to teach their students. "The Third International Mathematics and Science Study indicate that instructional mathematics practices in American schools tend to focus on fragmented procedures and topics rather than concepts that are central to continued learning of mathematics" ((Carlson, Oehrtman, & Engelke), National Center for Education Statistics).

The Pre-calculus Concept Assessment instrument is in hoped used as a model to reflect on the role of cognitive research and also provide a foundation to help students succeed in their first course of calculus. The first step in achieving this goal of developing an instrument is to review the literature and create studies.

Two forms of viewing a function: action views of a function and the process view of a function. They are both used in an order form. With the action view, students use symbolic manipulations and procedural techniques as a way to map input and output values. Once they can imagine that, then the students develop a process view and for those who are unable to have this development have difficulty manipulating functions and as a result solving word problems. Covariational reasoning is “essential for representing and interpreting the changing nature of quantities in a wide array of function situations and for understanding major concepts in beginning calculus” (Carlson, Oehrtman, & Engelke) which is also tied to the process view of functions.

There are two reasons why students have a weak understanding of functions. One is that students only see a function as being a curved on a plane instead of the set input and output values on the independent axis and dependent axis. Another reason is the students’ cannot grasp contextual relationships with symbols. Both of these factors are part of the students’ lack of using symbols to create an algebraic formula.

Students’ who have the gift of action view use computational reasoning or procedure to define a formula or view a formula as set of instructions while those who have the gift of process view are able to imagine the set values to a point where they can manipulate it: the non-visualizer versus the visualizer.

Covariational reasoning is what students use to interpret graphs and “[understand] the concept of limit, derivative, related rate problems, accumulation, and the Fundamental Theorem of Calculus” (Carlson, Oehrtman, & Engelke). It has also led to mental actions divided into five categories. These categories describe the abilities used to represent and interpret dynamic functions.

Mental action 1 students recognize two quantities in a problem that “change together in a dependency relationship”(pg. 6). A mental action 2 student can “imagine the direction of change of one quantity while imagining a change in the other quantity” (Carlson, Oehrtman, & Engelke). Students’ within the third category of mental actions use covariational reasoning is an advancement to coordinate the “amount of change” of one and imagining changes in another. Mental action 4 students’ “are able to attend to how the rate of change of one quantity with respect to the other, changes while imagining changes in one quantity” (Carlson, Oehrtman, & Engelke). The mental action five involves one quantity changing while imagining the other having continuous changes.

The goal from these studies is to find a basic and common agreement about the foundation on how to teach the main ideas of calculus before taking the actual course which are formed by the PCA Taxonomy. Teachers and students need a valid tool that is effective in pre-calculus that will help better prepare students for Calculus I.

The Taxonomy is not an inventory of pre-calculus courses but it does focus on reasoning and understandings that all students are required to have before entering the calculus course. Within Reasoning abilities there are three categories: process view of function, covariational reasoning, and computational abilities. The process of function category accepts one set of input values to produce output values. Covariational reasoning category consists of students being able to analyze the values and computational abilities deal with procedures and manipulations to solve problems.

Within the taxonomy are three categories to construct specific functions: understanding the meaning and usefulness of function evaluation, understanding patterns of functions, and understanding the meaning of the function representations and the connections within those representations.

There are four phases on reaching the PCA instrument. The first two consists of studies for the construction of the idea of Pre-Calculus and Calculus by interviewing students and re-wording questions to find distractions. The third phase is refining the Taxonomy and confirming multiple-choice problems and also within this phase are eight cycles which by the end of this phase the PCA has a worthy assessment along with the meaning of the constructs. During the final phase, these constructs (25-item version) is tested on a diverse population of students where the scores will determine “if the PCA is sensitive to instructional interventions” (Carlson, Oehrtman, & Engelke). On a side note, the focus was not on the scores, but if the students used reasoning abilities and understanding to solve the test and earn the score they received.

Through each phase, the main goal was to see if the assessment was a valid tool to determine whether students had the knowledge and ability of using pre-calculus to understand calculus. The emphasis should not be on the assessment itself, but actually using it as a valid instrument to know the students’ cognitive process that helps their learning ability.

The use of having a medium is a great instrument and also is helpful for humans to incorporate their thoughts. The concept of cognitive ecology and the theory of reorganization are very much similar where using computer system skills within human meditating is “the basis of this reorganization” (Villarreal, 2000) though the characterization of both concepts creates a gap.

With Borba and Villarreal’s research it has been recognized that even with the opportunity to use a graphing calculator to solve a problem, students still want to use paper and a pencil and without it, it would be difficult to solve any math problem algebraically. There is also the case that some students feel more “comfortable” using pencil and paper than using a computer which makes solving problems easier. “The use of the computer carrying out the fundamental role of experimentation, including the processes of proof is termed experimental mathematics” (Villarreal, 2000).

The computer is actually become a more technological way of visualization without rejecting the algebra. From Souza, the computers emphasize the visualization that “opens new options in the study of mathematics for those who are blocked with respect to algebra” (Villarreal, 2000).

Presmeg’s study of visualization, where students are either classified as visualizers or non-visualizers, points out that the math tests and curriculum favor the non-visualizers while the visualizers go unrecognized.

Within Villarreal’s study, there are two styles of approach: algebraic approach and visual approach.

The algebraic approach doesn’t necessarily need a computer because the process is carried out with paper and a pencil or through a mental process. According to Villarreal, the approach is characterized by: “

- a preference for analytic solutions when graphic solutions are also possible;
- difficulty in establishing graphic interpretations of analytic solutions;
- when a graphic solution is requested, a brief run through the analytic one is needed;
- facility with formulating conjectures and refutations or generating explanations based on formulas or equations” (Villarreal, 2000).

As for the visual approach, “the computer is used to verify conjectures” (Villarreal, 2000). Computers are used more to answer more graph problems, however they sometimes give confusion when it comes to points on the graph, hence the use of the algebraic methods to solve equations and the zoom button.

It is clear that algebraic and visual approach can be used for any student but it just depends on the problem itself, though some students prefer to use a pencil and paper to solve situations because the computer can create difficulties and challenges.

In conclusion, this article is supposed to set an equal balance between visual and algebraic approach in mathematics education. However, the computer challenges the techniques used in teaching calculus. The prototype of the algorithmic, also known as the computer, has actually helped this challenge in mathematics education.

Ecological effects, derived from Levy, are the emphasis and need for exact and single solutions along with limited value attributed to the visual and experimental in math education. These effects are from the process of solving mathematics with the use of technology. Technology in cognitive ecologies should be taken into consideration in re-thinking of math education.

Technological advances like graphing calculators and computer software have expanded the use of visualization as well as help other see what they could not within the science and mathematics field.

Mathematics explains technology and patterns even though visualization cannot help one see all of it. However, metaphorically speaking, Steen says, “if mathematics is the science of patterns, it is natural to try to find the most effective ways to visualize these patterns…" (Zimmermann & Cunningham)

The concept of visualization is "to form a mental image" (Zimmermann & Cunningham) but with the aid of tools such as a pencil and paper or computers. The use of visualization in mathematics is to provide a better understanding of mathematical situations. This process defers from vision because visualization requires for one to create an image in ones' mind which takes intuition.

A helpful facet to understanding visualization is computers because they can display graphics. Computer graphics is a recent aid in nonlinear systems and mathematical visualization but visualization is not just computer based; it can also be with a pencil, paper and the ability to draw.

Knowledge is part of the links to the purpose of teaching and for it to be successful, sequential presentations must be necessary. "Compartmentalization of knowledge occurs not only because it links between concepts and procedures are destroyed and omitted, but also because in school, knowledge is necessarily taught separate from its context" (Eisenberg and Dreyfus). In contrast to Chevallard's theory about visual processing versus analytic processing, Larkin and Simon claim that the two processes are different. However, teachers in school prefer to use sentential processing methods, in other words, analytic processing, over visual representations because "it is the most efficient way for teachers to present material to students" (Eisenberg & Dreyfus, On the Reluctance to Visualize in Mathematics).

In other efforts several have used software in order to incorporate more visual techniques. Artigue built computer software has become a useful tool to help students reduce the complexity of mathematical situations. Heid created similar software to perform manipulations.

There is the counterargument that leads to the question, “What if there are students that are visually impaired?” Honestly, hopefully technology can solve that problem. With advanced computer graphics, there may be a chance that using instant three dimensional models can help break this barrier, but such technology has not come to reality. Only the possibility of having pre-made models can help but with so many different visual aspects to solve mathematical problems, technology needs to advance even more into the future.

In conclusion, students have a hard time understanding visual frameworks and how to use visual thinking but graphing calculators and computer software helps students overcome this difficulty so as to grasp complicated problems.Visualization in the classroom can improve the understanding of mathematics for high school students. Though factors contributing to the decline of studying math is not completely known or understood, there are multiple events that happen to students that make them decide to stray away from pursuing a career in the STEM field. As mentioned before from Seymour, “poor teaching by faculty, course pace, course load, inadequate academic advising, financial problems, and language barriers” (Carlson, Oehrtman, & Engelke) can contribute to these events. However, for this decline to fade away, teachers need to incorporate more visual methods within the classroom. This practice of administering visualization more in math education could start in the early on stages of Algebra I and II. As a consequence, students may be encouraged to pursue a future in math.

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